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Invariant solutions of a supersymmetric fluid model

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Abstract
In this paper, we present a method for constructing invariant solutions of the supersymmetric Chaplygin gas in $(1+1)$ dimensions. This approach is based on the use of a generalized Legendre transformation, through which we transform the original field equations into a new set of equations involving the velocity and sound speed of the fluid as independent variables. We describe the Lie symmetry properties of the equations in both coordinate systems, and make use of a systematic subgroup classification to determine certain classes of group-invariant solutions of the transformed field equations. Where it is possible, the Legendre transformation is applied in reverse in order to obtain equivalent solutions of the standard field equations. A number of analytic solutions of the supersymmetric Chaplygin gas in one spatial dimension are found. In addition, certain basic elements of a possible extension of our method to the supersymmetric Chaplygin gas in two spatial dimensions have been formulated. In particular, the $(2+1)$-dimensional analogue of the transformed equation has been determined, and some of its Lie point symmetries have been identified. For a certain specific case, it has been demonstrated how an invariant solution can be inverted and then extended to a solution of the planar supersymmetric model.

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1. Introduction

A few years ago, a series of lectures were given by R Jackiw on the subject of fluid mechanics [1]. A number of topics were covered, including a general description of fluids with vanishing and non-vanishing vorticity and the properties of certain specific models. In particular, it was shown that two distinct parametrizations of the Nambu–Goto action for a d-brane evolving in a $(d+1, 1)$-dimensional target space–time lead respectively to two specific fluid dynamical models, namely the Galileo-invariant Chaplygin gas in d spatial dimensions and the Poincaré-invariant Born–Infeld model for a scalar in d spatial dimensions. The symmetry Lie algebras of the Chaplygin and Born–Infeld equations in one spatial dimension were systematically analysed, and a full classification of the one- and two-dimensional subalgebras was performed [2, 3]. A number of classes of invariant and partially invariant solutions were obtained.

In addition, a number of highly original theories of fluid mechanics were proposed by Jackiw *et al* [1]. These extensions of the classical theory were based primarily on the application of methods previously used in particle physics to the context of a classical field theory. Of special interest was the generalization of the Chaplygin gas models in one and two spatial dimensions to supersymmetric theories involving Grassmann (fermionic) variables. Unlike other supersymmetric extensions of fluid mechanics (see, for example, [4, 5]), these theories do not involve a superspace formulation. Nevertheless, the possibility of such a formulation will be considered briefly in section 2.

While the one-dimensional case [6] is the main focus of this paper, a number of important observations are made concerning the two-dimensional supersymmetric theory.

The primary purpose of this paper is to employ a generalized version of the Legendre transformation in order to determine invariant solutions of the supersymmetric Chaplygin gas in (1 + 1) dimensions. We want to put a new light on the equations of motion by solving them while taking into account their invariance properties in the previously obtained results [2]. Emphasis will be placed on the search for solutions through the formulation of the field equations in the new transformed coordinates. Let us mention that the Legendre transformation leads to a linearization of the equations only for the one-dimensional supersymmetric theory. In this case, the general solution can be obtained. Our aim in searching for special solutions is to understand deeply the connection between invariant solutions in both coordinate systems and eventually use them to fully extend our analysis to the supersymmetric model in (2 + 1) dimensions.

This paper is organized as follows. In section 2, we examine the field equations for the supersymmetric Chaplygin gas on a line, and present a method for constructing group-invariant solutions through a generalization of the Legendre transformation. Section 3 is devoted to a description of the Lie symmetry structure of both the standard equations of the (1 + 1)-dimensional supersymmetric Chaplygin gas (Lie superalgebra $\mathcal{G}_s$) and their transformed version (Lie algebra $\mathcal{L}$). We recall the classification of one-dimensional subalgebras of the bosonic sector of $\mathcal{G}_s$, and then perform the same analysis for the one-dimensional subalgebras of the finite sector of $\mathcal{L}$. In section 4, we make use of the classification of $\mathcal{L}$ in order to describe and discuss certain classes of group-invariant solutions of the transformed field equations. Where it is possible, the Legendre transformation is applied in reverse in order to obtain equivalent solutions of the standard field equations which are then extended to full solutions of the linear fermionic model. The connection between the latter and solutions obtained directly from the classification of $\mathcal{G}_s$ is examined. In section 5, we discuss how our analysis could be extended to the case of the supersymmetric Chaplygin gas on a plane, through the use of a generalized Legendre transformation in three independent variables. Invariant solutions of the planar supersymmetric model are obtained. Finally, section 6 contains observations and a discussion of further applications of our results.

2. Supersymmetric Chaplygin gas in one dimension

A supersymmetric generalization of the Chaplygin gas on a line was proposed in 2001 by Bergner and Jackiw [6]. The velocity of the fluid v is supplemented by a Grassmann variable or fermionic field $\psi(t, x)$ such that

$$v = \theta_x - \tfrac{1}{2}\psi\psi_x. \tag{1}$$

The equations of motion read as

$$\rho_t + \partial_x(\rho v) = 0, \tag{2}$$

$$\theta_t + v\theta_x = \frac{1}{2}v^2 + \frac{\lambda}{\rho^2} - \frac{\sqrt{2\lambda}}{2\rho}\psi\psi_x, \tag{3}$$

$$\psi_t + \left(v + \frac{\sqrt{2\lambda}}{\rho}\right)\psi_x = 0. \tag{4}$$

The continuity equation (2) and the Euler equation (3) are therefore modified from their original form in order to include the Grassmann variable ψ, while equation (4) is new. The velocity v satisfies the evolution equation

$$v_t + vv_x = \partial_x\left(\frac{\lambda}{\rho^2}\right). \tag{5}$$

Instead of using the density ρ, we make use of the sound speed variable

$$s = \frac{\sqrt{2\lambda}}{\rho}, \tag{6}$$

so that the equations of motion (2)–(4) become

$$s_t = -vs_x + sv_x, \tag{7}$$

$$\theta_t = -v\theta_x + \tfrac{1}{2}v^2 + \tfrac{1}{2}s^2 - \tfrac{1}{2}s\psi\psi_x, \tag{8}$$

$$\psi_t = -(v+s)\psi_x, \tag{9}$$

and equation (5) becomes

$$v_t + vv_x = ss_x. \tag{10}$$

It is to be noted that in the case where $\psi\psi_x = 0$, equations (7)–(10) are decoupled, and that solutions (θ, s) of the supersymmetric equations are simply those of the ordinary (bosonic) Chaplygin equations, while equation (9) becomes linear in ψ.

The significance of the modified equations (7), (9) and (10) may be considered in the context of another supersymmetric field theory. A supersymmetric generalization of the Chaplygin gas equations was constructed by Das and Popowicz [5] in the field of polytropic gas dynamics using a superspace formulation. However, it has been observed that choosing to supplement the bosonic variables (ρ, v) with superspace (fermionic) variables does not lead to a system of superequations that could be expanded to provide the set of field equations (2), (4) and (5). By choosing instead the bosonic variables (s, v), along with the set of equations (7), (9) and (10), it is possible to create an $N = 1$-superspace formulation for the field equations. Indeed, if we define the two fermionic fields

$$S(t,x;\vartheta) = \varphi_1(t,x) + \vartheta s(t,x), \qquad V(t,x;\vartheta) = \varphi_2(t,x) + \vartheta v(t,x), \tag{11}$$

for the $N = 1$-superspace $(t, x; \vartheta)$, where ϑ satisfies $\vartheta^2 = 0$, we may construct the superequations

$$S_t = -DVD^2S + DSD^2V, \qquad V_t = -DVD^2V + DSD^2S, \tag{12}$$

using the supercovariant derivative defined as $D = \partial_\vartheta + \vartheta\partial_x$. Separating the fermionic and bosonic components of the superequations (12), we easily recover equations (7) and (10) for the bosonic variables while, for the fermionic variables, we obtain

$$\varphi_{1,t} = -v\varphi_{1,x} + s\varphi_{2,x}, \qquad \varphi_{2,t} = -v\varphi_{2,x} + s\varphi_{1,x}. \tag{13}$$

It is evident that equation (9) will be recovered if we take $\psi = \varphi_1 - \varphi_2$. Such a system (12) refers in fact to a so-called SUSY-B extension of the Chaplygin gas [5].

Let us recall that the Legendre transformation has been performed on the bosonic Chaplygin gas equations (i.e. when $\psi = 0$) in order to linearize them and obtain the general solution [1]. Using this result as an inspiration, we extend the method to the supersymmetric case. We begin with the dependent variables

$$q = \theta_t - \tfrac{1}{2}\psi\psi_t, \qquad v = \theta_x - \tfrac{1}{2}\psi\psi_x. \tag{14}$$

Using equations (8) and (9) to substitute the values of θ_t and ψ_t into q, we obtain the following relationship which links s to v and q:

$$s^2 = 2q + v^2. \tag{15}$$

Later, it will be convenient to use the (s, v) coordinate system instead of (q, v). Equation (7), when expressed in terms of v and q then becomes

$$\frac{\partial q}{\partial t} + v\left(\frac{\partial v}{\partial t} + \frac{\partial q}{\partial x}\right) - 2q\frac{\partial v}{\partial x} = 0. \tag{16}$$

We modify equation (16) to a differential equation for x and t in terms of v and q. Indeed, the change of variables $(t, x) \leftrightarrow (q, v)$ implies that

$$\frac{\partial x}{\partial q} = -\frac{1}{J}\frac{\partial v}{\partial t}, \qquad \frac{\partial x}{\partial v} = \frac{1}{J}\frac{\partial q}{\partial t}, \qquad \frac{\partial t}{\partial q} = \frac{1}{J}\frac{\partial v}{\partial x}, \qquad \frac{\partial t}{\partial v} = -\frac{1}{J}\frac{\partial q}{\partial x}, \tag{17}$$

where $J = \partial(q, v)/\partial(t, x)$ is the Jacobian of the change of variables. Equation (16) is then transformed to

$$-\frac{\partial x}{\partial v} + v\left(\frac{\partial x}{\partial q} + \frac{\partial t}{\partial v}\right) + 2q\frac{\partial t}{\partial q} = 0. \tag{18}$$

By analogy with the ordinary case, we postulate the existence of a function $W = W(q, v)$ such that

$$t = W_q, \qquad x = W_v. \tag{19}$$

Equation (18) can be rewritten through relations (19) as

$$W_{vv} - 2vW_{qv} - 2qW_{qq} = 0. \tag{20}$$

We now change to the coordinates (s, v). Using (15) and writing $W(q, v) = W(s, v)$, we finally get the equation

$$W_{vv} - W_{ss} + \frac{2}{s}W_s = 0, \tag{21}$$

for which the general solution is given by Jackiw [1]:

$$W(v, s) = f(v + s) - sf'(v + s) + g(v - s) + sg'(v - s), \tag{22}$$

where f and g are arbitrary functions in their respective arguments.

We must now incorporate the fermionic field $\psi(t, x)$ into this framework. We begin by expressing ψ in terms of the coordinates s and v:

$$\psi(t, x) = \psi(t(s, v), x(s, v)) = \chi(s, v). \tag{23}$$

Using the change of variables in the derivatives given by

$$\partial_t = s_t\partial_s + v_t\partial_v, \qquad \partial_x = s_x\partial_s + v_x\partial_v, \tag{24}$$

and the fact that s_t and v_t may be deduced from (7) and (10), we deduce the equation on χ from equation (9). Indeed, we get

$$(s_x + v_x)(\chi_s + \chi_v) = 0, \tag{25}$$

which implies that

$$\chi_s + \chi_v = 0, \tag{26}$$

if $s_x + v_x \neq 0$. The general solution is therefore

$$\chi(s, v) = \phi(v - s), \tag{27}$$

where ϕ is an arbitrary function of $v - s$. This has been deduced directly by Jackiw [1]. Let us show that the condition $s_x + v_x \neq 0$ is in fact redundant if we take into account the form (22) of the general solution W. Indeed,

$$s_x + v_x = 0 \quad \leftrightarrow \quad s(W_{sv} - W_{ss}) + W_s = 0, \qquad (s \neq 0), \tag{28}$$

since we can write (from (15), (17) and (19))

$$s_x = \frac{J}{s}\left(v\frac{\partial t}{\partial q} - \frac{\partial t}{\partial v}\right) = -\frac{J}{s^2}W_{sv}, \tag{29}$$

$$v_x = J\frac{\partial t}{\partial q} = \frac{J}{s^3}(sW_{ss} - W_s). \tag{30}$$

Inserting $W = (22)$ into (28), we get $g''' = 0$. Using $g(v-s) = a(v-s)^2 + b(v-s) + c$, the original variables t and x may be written (from (19) and (22)) as

$$t = -f''(v+s) - 2a, \qquad x = f'(v+s) - (v+s)f''(v+s) + b, \tag{31}$$

showing that the change of variables $(t, x) \leftrightarrow (s, v)$ is not invertible.

3. Structure of the symmetry Lie superalgebra

3.1. The standard form of the field equation

The Lie algebra $\mathcal{G}$ of the supersymmetric Chaplygin gas equations (2)–(4) in one spatial dimension is spanned by the six independent vector fields [2, 7]:

$$\begin{aligned} &P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad B = t\partial_x + x\partial_\theta, \qquad Z = \partial_\theta, \\ &D_1 = 2t\partial_t + x\partial_x + \rho\partial_\rho, \qquad D_2 = x\partial_x + 2\theta\partial_\theta - \rho\partial_\rho + \psi\partial_\psi. \end{aligned} \tag{32}$$

Here, P_0 and P_1 represent translations in the independent variables t and x respectively, B consists of a Galilean boost, Z corresponds to a shift in the potential θ, and D_1 and D_2 are dilations in the dependent and independent variables. In addition, we have the following two supersymmetries which link the bosonic and fermionic variables:

$$Q = \psi\partial_x - \rho\psi_x\partial_\rho + \left(\frac{1}{2}\psi\theta_x - \frac{\sqrt{2\lambda}}{2\rho}\psi\right)\partial_\theta + \left(\frac{1}{2}\psi\psi_x - \theta_x + \frac{\sqrt{2\lambda}}{\rho}\right)\partial_\psi, \tag{33}$$

and

$$\tilde{Q} = -\tfrac{1}{2}\psi\partial_\theta - \partial_\psi. \tag{34}$$

These transformations correspond to the conserved charges and supercharges identified by Jackiw and Bergner [1, 6], and can be derived by the use of Noether's theorem. It should be mentioned that the vector field $\tilde{Q}$ essentially produces a translation of the dependent fermionic variable, i.e. $\psi' = \psi + \tilde{\eta}$ for $\tilde{\eta}$ a fermionic constant, while the vector field Q implies a transformation mixing dependent and independent variables, i.e. $x' = x + \eta\psi$, where again η is a fermionic constant. This last supersymmetric transformation also leads

Table 1. Supercommutation table for the Lie superalgebra $\mathcal{G}_s$.

X\Y	D_1	D_2	B	Z	P_1	P_0	Q	$\tilde{Q}$
D_1	0	0	B	0	$-P_1$	$-2P_0$	$-Q$	0
D_2	0	0	$-B$	$-2Z$	$-P_1$	0	0	$-\tilde{Q}$
B	$-B$	B	0	0	$-Z$	$-P_1$	$\tilde{Q}$	0
Z	0	$2Z$	0	0	0	0	0	0
P_1	P_1	P_1	Z	0	0	0	0	0
P_0	$2P_0$	0	P_1	0	0	0	0	0
Q	Q	0	$-\tilde{Q}$	0	0	0	$2P_0$	$-P_1$
$\tilde{Q}$	0	$\tilde{Q}$	0	0	0	0	$-P_1$	Z

to $\psi' = \psi + \eta(s - v)$ and could probably be related to the transformation included in the superspace formulation (12), i.e. $\mathcal{Q} = \partial_\vartheta - \vartheta\partial_x$.

The vector fields (32)–(34) therefore generate a Lie superalgebra $\mathcal{G}_s$ whose supercommutation relations are given in table 1. It should be noted that these relations consist of commutators

$$[A, B] = AB - BA, \tag{35}$$

in the case where at least one of A and B is even (bosonic), and of anticommutators

$$\{A, B\} = AB + BA, \tag{36}$$

in the case where both A and B are odd (fermionic).

The classification of subalgebras of the bosonic sector $\mathcal{G}$ of $\mathcal{G}_s$ has already been performed [2, 3]. We are interested here only in one-dimensional such subalgebras. There are the splitting subalgebras

$$\begin{gathered}\mathcal{L}_1 = \{D_1\}, \qquad \mathcal{L}_2 = \{D_2\}, \qquad \mathcal{L}_3 = \{B\}, \qquad \mathcal{L}_{4,a} = \{D_1 + aD_2, a \in \mathbb{R}\backslash\{0\}\}, \\ \mathcal{L}_{5,\varepsilon} = \{D_1 + D_2 + \varepsilon B, \varepsilon = \pm 1\}, \qquad \mathcal{N}_1 = \{P_0\}, \qquad \mathcal{N}_2 = \{P_1\}, \\ \mathcal{N}_3 = \{Z\}, \qquad \mathcal{N}_{4,\varepsilon} = \{Z + \varepsilon P_0, \varepsilon = \pm 1\},\end{gathered} \tag{37}$$

and the non-splitting ones

$$\begin{gathered}\mathcal{K}_{1,\varepsilon} = \{D_1 + \varepsilon Z, \varepsilon = \pm 1\}, \qquad \mathcal{K}_{2,\varepsilon} = \{D_1 - D_2 + \varepsilon P_1, \varepsilon = \pm 1\}, \\ \mathcal{K}_{3,\varepsilon} = \{D_2 + \varepsilon P_0, \varepsilon = \pm 1\}, \qquad \mathcal{K}_{4,\varepsilon} = \{B + \varepsilon P_0, \varepsilon = \pm 1\}.\end{gathered} \tag{38}$$

All these subalgebras, except $\mathcal{N}_3$ which does not contain any derivative with respect to the independent variables t and x, have been used to construct specific invariant solutions of equations (2) and (3) in the bosonic case. [2]. Generators of supersymmetries like (33) and (34) cannot be included because they give rise by anticommutation to a bosonic generator, leading thus to a two-dimensional subalgebra. For instance, any subsuperalgebra containing the generator Q must necessarily also contain P_0. Consequently, any solution invariant under Q will belong to the class of invariant solutions corresponding to subalgebra $\mathcal{N}_1$, and will therefore not be fundamentally distinguishable from solutions obtained from $\mathcal{N}_1$.

3.2. The transformed equation

Let us now consider the Lie algebra $\mathcal{L}$ of the transformed equation (21). It is easy to show [8] that this algebra is spanned by the generators:

$$\begin{gathered}D = s\partial_s + v\partial_v + W\partial_W, \qquad V = \partial_v, \qquad C = sv\partial_s + \tfrac{1}{2}(s^2 + v^2)\partial_v + vW\partial_W, \\ M = W\partial_W, \qquad S_\beta = \beta(s, v)\partial_W,\end{gathered} \tag{39}$$

where the function β satisfies equation (21). It should be noted that, in contrast to the Lie algebra $\mathcal{G}$ in the case of the standard equations, $\mathcal{L}$ contains an infinite-dimensional family of symmetries S_β. This allows us to extend our invariance analysis to a much greater class of subalgebras, and therefore potentially obtain additional invariant solutions of the one-dimensional supersymmetric model. Since the system of transformed equations (21) and (26) is decoupled and moreover the equation for χ is linear, the symmetries of equation (21) can be analysed separately.

The superalgebra $\mathcal{L}_s$ of the complete system of transformed equations (21) and (26) contains the symmetries given in (39) in addition to the following generators which involve the fermionic field χ. We have two families of fermionic generators (supersymmetries):

$$\mathcal{Q}_\alpha = \alpha(v-s)\partial_\chi, \qquad \mathcal{R}_\omega = \omega(v-s)\chi\partial_\chi, \tag{40}$$

where α is an arbitrary bosonic function of $v-s$, and ω is an arbitrary fermionic function of $v-s$. We also have their bosonic counterparts:

$$\mathcal{A}_\lambda = \lambda(v-s)\partial_\chi, \qquad \mathcal{B}_\kappa = \kappa(v-s)\chi\partial_\chi, \tag{41}$$

where λ is now fermionic, and κ bosonic. The supercommutation relations of the superalgebra $\mathcal{L}_s$ are summarized in table 2. In addition, for any two solutions β_1 and β_2 of equation (21), $\left[S_{\beta_1}, S_{\beta_2}\right] = 0$, and we also have $\left[\mathcal{A}_{\lambda_1}, \mathcal{A}_{\lambda_2}\right] = 0$, $\left[\mathcal{B}_{\kappa_1}, \mathcal{B}_{\kappa_2}\right] = 0$, $\left\{\mathcal{Q}_{\alpha_1}, \mathcal{Q}_{\alpha_2}\right\} = 0$, $\left\{\mathcal{R}_{\omega_1}, \mathcal{R}_{\omega_2}\right\} = 0$. Since the generators $\mathcal{A}_\lambda$, $\mathcal{B}_\kappa$, $\mathcal{Q}_\alpha$ and $\mathcal{R}_\omega$ do not involve derivatives with respect to the independent variables s and v, they cannot be used to obtain invariant solutions through the classical method of symmetry reduction. We therefore concentrate on the bosonic sector $\mathcal{L}$ of the algebra rather than on the complete superalgebra $\mathcal{L}_s$.

In order to classify the subalgebras, we decompose the structure of $\mathcal{L}$ into the following form:

$$\mathcal{L} = \{\{D, V, C\} \oplus \{M\}\} \quad \Subset_\beta \quad \{S_\beta\}. \tag{42}$$

Concentrating once again on the one-dimensional subalgebras, we perform the classification in three steps [9].

(i) The simple algebra $\mathcal{F} = \{D, V, C\}$ is isomorphic to the Lie algebra $o(2,1)$ whose subalgebra classification is known [9]. Its one-dimensional subalgebras are

$$\mathcal{F}_{1,1} = \{D\}, \qquad \mathcal{F}_{1,2} = \{V\}, \qquad \mathcal{F}_{1,3} = \{2V + C\}. \tag{43}$$

(ii) Next, we consider the direct sum

$$\mathcal{S} = \{\{D, V, C\} \oplus \{M\}\}. \tag{44}$$

The one-dimensional splitting subalgebras of $\mathcal{S}$ are

$$\mathcal{S}_1 = \{M\}, \qquad \mathcal{S}_2 = \{D\}, \qquad \mathcal{S}_3 = \{V\}, \qquad \mathcal{S}_4 = \{2V + C\}. \tag{45}$$

In addition, we obtain the following non-splitting subalgebras of $\mathcal{S}$:

$$\begin{aligned} &\mathcal{S}_{5,a} = \{D + aM, a \neq 0\}, \qquad \mathcal{S}_{6,\varepsilon} = \{V + \varepsilon M, \varepsilon = \pm 1\}, \\ &\mathcal{S}_{7,a} = \{2V + C + aM, a \neq 0\}. \end{aligned} \tag{46}$$

(iii) The complete Lie algebra $\mathcal{L}$ is constructed through the semi-direct sum of the previously considered algebra $\mathcal{S}$ with the infinite-dimensional Lie algebra spanned by the generators S_β. Each solution $\beta(s, v)$ of equation (21) imposes a different structure on the further classification of the subalgebras of the semi-direct sum. We will consider several instances of this.

The usefulness of the classification is demonstrated in the fact that it allows us to find all corresponding symmetry reductions of equation (21) under the classified non-equivalent subalgebras of the symmetry algebra $\mathcal{S}$.

Table 2. Supercommutation table for the Lie superalgebra $\mathcal{L}_s$ spanned by the vector fields (39).

X\Y	D	V	C	M	S_β	$\mathcal{A}_\lambda$	$\mathcal{B}_\kappa$	$\mathcal{Q}_\alpha$	$\mathcal{R}_\omega$
D	0	$-V$	C	0	$S_{(s\beta_s+v\beta_v-\beta)}$	$\mathcal{A}_{(v-s)\lambda'}$	$\mathcal{B}_{(v-s)\kappa'}$	$\mathcal{Q}_{(v-s)\alpha'}$	$\mathcal{R}_{(v-s)\omega'}$
V	V	0	D	0	S_{β_v}	$\mathcal{A}_{\lambda'}$	$\mathcal{B}_{\kappa'}$	$\mathcal{Q}_{\alpha'}$	$\mathcal{R}_{\omega'}$
C	$-C$	$-D$	0	0	$S_{(sv\beta_s-v\beta+\frac{(s^2+v^2)\beta_v}{2})}$	$\mathcal{A}_{\frac{(v-s)^2\lambda'}{2}}$	$\mathcal{B}_{\frac{(v-s)^2\kappa'}{2}}$	$\mathcal{Q}_{\frac{(v-s)^2\alpha'}{2}}$	$\mathcal{R}_{\frac{(v-s)^2\omega'}{2}}$
M	0	0	0	0	$-S_\beta$	0	0	0	0
S_β	$S_{(\beta-s\beta_s-v\beta_v)}$	$-S_{\beta_v}$	$S_{(v\beta-sv\beta_s-\frac{(s^2+v^2)\beta_v}{2})}$	S_β	0	0	0	0	0
$\mathcal{A}_\lambda$	$\mathcal{A}_{(s-v)\lambda'}$	$-\mathcal{A}_{\lambda'}$	$-\mathcal{A}_{\frac{(v-s)^2\lambda'}{2}}$	0	0	0	$\mathcal{A}_{\lambda\kappa}$	0	$-\mathcal{Q}_{\lambda\omega}$
$\mathcal{B}_\kappa$	$\mathcal{B}_{(s-v)\kappa'}$	$-\mathcal{B}_{\kappa'}$	$-\mathcal{B}_{\frac{(v-s)^2\kappa'}{2}}$	0	0	$-\mathcal{A}_{\lambda\kappa}$	0	$-\mathcal{Q}_{\alpha\kappa}$	0
$\mathcal{Q}_\alpha$	$\mathcal{Q}_{(s-v)\alpha'}$	$-\mathcal{Q}_{\alpha'}$	$-\mathcal{Q}_{\frac{(v-s)^2\alpha'}{2}}$	0	0	0	$\mathcal{Q}_{\alpha\kappa}$	0	$-\mathcal{A}_{\alpha\omega}$
$\mathcal{R}_\omega$	$\mathcal{R}_{(s-v)\omega'}$	$-\mathcal{R}_{\omega'}$	$-\mathcal{R}_{\frac{(v-s)^2\omega'}{2}}$	0	0	$\mathcal{Q}_{\lambda\omega}$	0	$-\mathcal{A}_{\alpha\omega}$	0

Table 3. Invariants of the one-dimensional subalgebras of $\mathcal{S}$.

Subalgebra	Symmetry variable(s)	Function W
$\{M\}$	$\xi = s, v$	No function
$\{D\}$	$\xi = \frac{v}{s}$	$W = sF(\xi)$
$\{V\}$	$\xi = s$	$W = F(\xi)$
$\{2V + C\}$	$\xi = \frac{s^2-v^2-4}{s}$	$W = sF(\xi)$
$\{D + aM, a \neq 0\}$	$\xi = \frac{v}{s}$	$W = s^{a+1}F(\xi)$
$\{V + \varepsilon M, \varepsilon = \pm 1\}$	$\xi = s$	$W = \mathbf{e}^{\varepsilon v}F(\xi)$
$\{2V + C + aM, a \neq 0\}$	$\xi = \frac{s^2-v^2-4}{s}$	$W = s\,\mathbf{e}^{\frac{a}{2}\tan^{-1}\left(\frac{v^2-s^2-4}{4v}\right)}F(\xi)$

Table 4. Reduced equations and solutions $W(s, v)$ obtained from one-dimensional subalgebras of $\mathcal{S}$(K_0 and K_1 are arbitrary constants).

Subalgebra	Reduced equation(s)	Solution $W(s, v)$
$\{D\}$	$(1-\xi^2)F_{\xi\xi} - 2\xi F_\xi + 2F = 0$	$K_0 v + K_1 s - \frac{1}{2}K_1 v \ln\left(\frac{s+v}{s-v}\right)$
$\{V\}$	$\frac{2}{\xi}F_\xi - F_{\xi\xi} = 0$	$\frac{1}{3}K_0 s^3 + K_1$
$\{2V + C\}$	$(\xi^2 + 16)F_{\xi\xi} + 2\xi F_\xi - 2F = 0$	$K_0(s^2 - v^2 - 4) + K_1\left(s + \frac{1}{4}(s^2 - v^2 - v^2 - 4)\tan^{-1}\left(\frac{s^2-v^2-4}{4s}\right)\right)$
$\{D - M\}$	$(1-\xi^2)F_{\xi\xi} - 4\xi F_\xi = 0$	$K_0\left(\ln\left(\frac{s+v}{s-v}\right) + \frac{2vs}{s^2-v^2}\right) + K_1$
$\{D + M\}$	$(1-\xi^2)F_{\xi\xi} + 2F = 0$	$K_0(s^2 - v^2) + K_1\left(vs + \frac{1}{2}(s^2 - v^2)\ln\left(\frac{v+s}{v-s}\right)\right)$
$\{D + aM\}_{(a\neq -1,0,1)}$	$(1-\xi^2)F_{\xi\xi} + 2(a-1)\xi F_\xi - (a+1)(a-2)F = 0$	$K_0(v-s)^a(v+as) + K_1(v+s)^a(v-as)$
$\{V + \varepsilon M\}_{(\varepsilon=\pm 1)}$	$F_{\xi\xi} - \frac{2}{\xi}F_\xi - F = 0$	$K_0(s-1)\,\mathbf{e}^{(\varepsilon v+s)} + K_1(s+1)\,\mathbf{e}^{(\varepsilon v-s)}$
$\{2V + C + aM\}_{(a\neq 0)}$	$(\xi^2+16)^2F_{\xi\xi} + 2\xi(\xi^2+16)F_\xi - 2(\xi^2 + 16 + 2a^2)F = 0$	$\mathbf{e}^{\frac{a}{2}\tan^{-1}\left(\frac{v^2-s^2-4}{4v}\right)}\Big[K_0(s^2 - v^2 - 4 + 2sa) \times \left(\frac{(-v+s+2\mathrm{i})(v+s+2\mathrm{i})}{(v+s-2\mathrm{i})(v-s+2\mathrm{i})}\right)^{\frac{a\mathrm{i}}{4}} + K_1(v^2 - s^2 + 4 + 2sa)\left(\frac{(v+s-2\mathrm{i})(v-s+2\mathrm{i})}{(-v+s+2\mathrm{i})(v+s+2\mathrm{i})}\right)^{\frac{a\mathrm{i}}{4}}\Big]$

4. Group-invariant solutions

4.1. Solutions of the bosonic model

In this section, we proceed to describe the solutions of the transformed differential equation (21) which are invariant under the one-dimensional subalgebras of $\mathcal{S}$. For each conjugacy class given above, we evaluate the invariants of the corresponding Lie subalgebra, and also the corresponding reduced differential equation. From the solution of each reduced equation, we obtain the respective solution $W(s, v)$ of the transformed differential equation (21), and the results are summarized in tables 3 and 4. In addition, we also consider certain subalgebras of $\mathcal{L}$ involving generators of type S_β.

Where it is possible, the Legendre transformation is applied in reverse in order to obtain the solutions $\theta(t, x)$, $\rho(t, x)$ and $\psi(t, x)$ of the original Chaplygin equations. We may proceed directly from the system of coordinates $(s, v, W(s, v))$ to the standard system of coordinates $(t, x, \theta(t, x))$ through the following procedure. We first determine t and x as functions of s and v through the relations

$$t = -\frac{1}{s}W_s, \qquad x = -W_v - \frac{v}{s}W_s. \tag{47}$$

Table 5. Bosonic solutions obtained from one-dimensional subalgebras of $\mathcal{S}$.

Subalgebra	Solution $\theta(t,x)$, $\rho(t,x)$	Link with the algebra $\mathcal{G}$
$\{V\}$	$\theta(t,x) = \frac{x^2}{2t} + \frac{t^3}{6K_0^2} + K_1$	$\{B\}$
	$\rho(t,x) = -\frac{\sqrt{2\lambda}K_0}{t}$	
$\{D\}$	$\theta(t,x) = -\frac{K_1^2}{2t}\cosh^2\left(\frac{1}{K_1}(x+K_0)\right)$	$\{D_1 - D_2\}$
	$\rho(t,x) = -\frac{\sqrt{2\lambda}t}{K_1\cosh^2\left(\frac{1}{K_1}(x+K_0)\right)}$	
$\{D - M\}$	$\theta(t,x) = K_0\ln\left(\frac{x^2+4K_0t}{x^2-4K_0t}\right) + K_1$	$\{D_1\}$
	$\rho(t,x) = \sqrt{2\lambda}\left(\frac{16K_0^2t^2-x^4}{4K_0x^3}\right)$	
$\{D + M\}$	$\theta(t,x) = -\frac{x^2}{4K_1}\frac{\left(1+\mathrm{e}^{K_1^{-1}(t+2K_0)}\right)}{\left(1-\mathrm{e}^{K_1^{-1}(t+2K_0)}\right)}$	$\{D_2\}$
	$\rho(t,x) = -\frac{2K_1\sqrt{2\lambda}}{x}$	
$\{D + 2M\}$	$\theta(t,x) = \frac{K_1}{108K_0^2}\frac{(12K_0x-t^2)^{3/2}}{\sqrt{4K_0^2-K_1^2}} + \frac{xt}{6K_0} - \frac{t^3}{108K_0^2}$	$\{D_1 + 3D_2\}$, $\{B + \varepsilon P_0\}$
	$\rho(t,x) = 3\sqrt{2\lambda}\frac{\sqrt{4K_0^2-K_1^2}}{\sqrt{12K_0x-t^2}}$	

This change of variables can in principle be inverted (provided that the Jacobian J does not vanish), and we obtain s and v as functions of t and x. The solution $\rho(t,x)$ is directly obtained through the relation

$$\rho(t,x) = \frac{\sqrt{2\lambda}}{s(t,x)}, \tag{48}$$

and the function $\theta(t,x)$ is determined by

$$\theta(t,x) = W(s,v) + xv + \tfrac{1}{2}t(s^2 - v^2). \tag{49}$$

The solutions $\theta(t,x)$ constructed through the above procedure are compared with those determined directly in [2] from the subgroup structure of the standard equations. The results are presented in table 5. In the case of subalgebra $\{D + 2M\}$, we obtain an equivalence with two distinct subalgebras, $\{D_1 + 3D_2\}$ and $\{B + \varepsilon P_0\}$. This occurs because the associated solution is in fact invariant under the two-dimensional subalgebra $\{D_1 + 3D_2, B + \varepsilon P_0\}$, which indicates that the two distinct one-dimensional subalgebras generate the same orbit.

Let us demonstrate the procedure through the following example. For the subalgebra $\{D + M\}$, the general solution of the transformed equation (21) is

$$W(s,v) = K_0(s^2 - v^2) + K_1\left(vs + \frac{1}{2}(s^2 - v^2)\ln\left(\frac{v+s}{v-s}\right)\right). \tag{50}$$

From relations (47), we get the transformation

$$t = -2K_0 - K_1\ln\left(\frac{v+s}{v-s}\right), \qquad x = -2K_1 s, \tag{51}$$

which may be inverted to give

$$v = \frac{1}{2K_1}x\frac{\left(1+\mathrm{e}^{-\frac{1}{K_1}(t+2K_0)}\right)}{\left(1-\mathrm{e}^{-\frac{1}{K_1}(t+2K_0)}\right)}, \qquad s = -\frac{1}{2K_1}x. \tag{52}$$

Equation (49) then yields the solution

$$\theta(t,x) = -\frac{x^2}{4K_1}\frac{\left(1+\mathbf{e}^{K_1^{-1}(t+2K_0)}\right)}{\left(1-\mathbf{e}^{K_1^{-1}(t+2K_0)}\right)}, \tag{53}$$

which is of the same form as the solution invariant under the subalgebra $\{D_2\}$ of $\mathcal{G}$, as expected from the fact that the change of variables (51) transforms the differential operator $s\partial_s + v\partial_v$ to $x\partial_x$.

4.2. *Additional subalgebras of* $\mathcal{S}$

We now return to the discussion of the solution for the subalgebra $\{D + aM\}$ where $a \neq 0, \pm 1, 2$. The solution in (s, v) space is given by

$$W(s,v) = K_0(v-s)^a(v+as) + K_1(v+s)^a(v-as), \tag{54}$$

so that the transformations to the (t, x) coordinate system are given by

$$\begin{aligned} t &= a(a+1)(K_0(v-s)^{a-1} + K_1(v+s)^{a-1}),\\ x &= (a^2-1)(K_0(v-s)^a + K_1(v+s)^a), \end{aligned} \tag{55}$$

which is not easy to invert in the general case. The corresponding solution $\theta(t, x)$ may, however, be written in terms of the variables v and s:

$$\theta(t,x) = \tfrac{1}{2}a(a-1)(K_0(v-s)^{a+1} + K_1(v+s)^{a+1}). \tag{56}$$

If we proceed to use the change of variables (55) to transform the differential operator $s\partial_s + v\partial_v$ in a manner analogous to that performed for the case $\{D + M\}$ above, we find

$$s\partial_s + v\partial_v \rightarrow (a-1)t\partial_t + ax\partial_x. \tag{57}$$

This is equivalent to the independent variable terms in the vector field $((a-1)/2)D_1 + ((a+1)/2)D_2$ in the (t, x) space. Thus we make the correspondence

$$D + aM \quad \leftrightarrow \quad (a-1)D_1 + (a+1)D_2. \tag{58}$$

This is consistent with the results given in table 5. Indeed, the cases $a = 0, -1, 1, 2$ respectively link subalgebras $\{D\}, \{D - M\}, \{D + M\}, \{D + 2M\}$ in (s, v) space with subalgebras $\{D_1 - D_2\}, \{D_1\}, \{D_2\}$ and $\{D_1 + 3D_2\}$ in (t, x) space.

Let us mention in particular the new case where $a = 1/2$. The invariants are

$$\xi = \frac{v}{s}, \qquad s^{-3/2}W, \tag{59}$$

so that the solution $W(s, v)$ of equation (21) is of the form

$$W = K_0(v-s)^{1/2}\left(v + \tfrac{1}{2}s\right) + K_1(v+s)^{1/2}\left(v - \tfrac{1}{2}s\right). \tag{60}$$

From relations (47), we get the transformation

$$x = -\tfrac{3}{4}K_0(v-s)^{1/2} - \tfrac{3}{4}K_1(v+s)^{1/2}, \qquad t = \tfrac{3}{4}K_0(v-s)^{-1/2} + \tfrac{3}{4}K_1(v+s)^{-1/2}. \tag{61}$$

The Jacobian J is given by

$$J = -\frac{9K_0K_1s}{16(v-s)^{3/2}(v+s)^{3/2}}, \tag{62}$$

and so the constants K_0 and K_1 must both be non-zero if solution (60) is to be invertible through the Legendre transformation. In the specific case where $K_0 = K_1$ we can invert relations (61) to obtain

$$s = \frac{4x}{9K_0^2t}\sqrt{9K_0^2xt + 4x^2t^2}, \qquad v = \frac{8x^2}{9K_0^2} + \frac{x}{t}. \tag{63}$$

The corresponding solution is therefore given by

$$\theta(t,x) = \frac{x^2}{2t} + \frac{8x^3}{27K_0^2}, \qquad \rho(t,x) = \frac{9\sqrt{2\lambda}K_0^2 t}{4x\sqrt{9K_0^2 xt + 4x^2t^2}}. \tag{64}$$

This corresponds to a solution invariant under the subalgebra $\{D_1 - 3D_2\}$ of $\mathcal{G}$, which has not been given before.

For the subalgebra $\{V + \varepsilon M\}$, the solution is given by

$$W(s,v) = K_0(s-1)\,\mathbf{e}^{(\varepsilon v+s)} + K_1(s+1)\,\mathbf{e}^{(\varepsilon v-s)}. \tag{65}$$

The transformations linking the (t, x) and (s, v) coordinate systems are given by

$$\begin{aligned} t &= -K_0\,\mathbf{e}^{s+\varepsilon v} + K_1\,\mathbf{e}^{-s+\varepsilon v}, \\ x &= -K_0(v + (s-1)\varepsilon)\,\mathbf{e}^{s+\varepsilon v} + K_1(v - (s+1)\varepsilon)\,\mathbf{e}^{-s+\varepsilon v}. \end{aligned} \tag{66}$$

Since the Jacobian cannot be allowed to vanish, the constants K_0 and K_1 are required to be nonzero. Even in the simplest case where $K_0 = K_1$, the change of variables (66) is difficult to invert. In terms of the variables s and v, the corresponding solution $\theta(t, x)$ can be written as

$$\begin{aligned} \theta(t,x) = {} & K_0\big((1-\varepsilon v)(s-1) - \tfrac{1}{2}(v^2+s^2)\big)\,\mathbf{e}^{(\varepsilon v+s)} \\ & + K_1\big((1-\varepsilon v)(s+1) + \tfrac{1}{2}(v^2+s^2)\big)\,\mathbf{e}^{(\varepsilon v-s)}. \end{aligned} \tag{67}$$

If we proceed to use the change of variables (66) to transform the differential operator, we obtain

$$\partial_v \to \varepsilon t\,\partial_t + (\varepsilon x + t)\,\partial_x \tag{68}$$

$$= t\partial_x + \varepsilon(x\partial_x + t\partial_t). \tag{69}$$

This is equivalent to the independent variable terms in the vector field $B + \frac{1}{2}\varepsilon(D_1 + D_2)$ in the (t, x) space, which is similar but not identical to the subalgebra $\{D_1 + D_2 + \varepsilon B\}$ identified in (37). The solution corresponding to this particular subalgebra had not been found previously.

For the subalgebras $\{2V + C\}$, and $\{2V + C + aM\}$, the change of variables (47) is difficult to invert, and corresponding solutions of the field equations in (t, x) space have not been found. In the case of $\{2V + C\}$, we require that K_1 be non-zero in order to obtain a nonvanishing Jacobian. For $\{2V + C + aM\}$, both K_0 and K_1 must be different from zero.

4.3. Subalgebras involving S_β

The question arises as to whether additional solutions of the field equations (2)–(4) may be found from subalgebras involving generators of the form S_β. Let us observe the effect of applying group conjugation by such an element to the generator $V = \partial_v$. If we define

$$Y = kS_\beta = k\beta(s,v)\partial_W, \tag{70}$$

where k is a constant, then the commutator terms in the Campbell–Baker–Hausdorff formula read as

$$[Y,V] = -kS_{\beta_v}, \qquad [Y,[Y,V]] = 0, \qquad [Y,[Y,[Y,V]]] = 0, \ldots, \tag{71}$$

so that the generator V is conjugate to $V - kS_{\beta_v}$. Thus, in order to obtain a new solution which is not a trivial extension of the solution already found for V, we must consider the subalgebra spanned by an element of the form $V + S_\alpha$, where $\alpha(s, v)$ obeys the following conditions:

(i) α is a solution of equation (21),
(ii) there does not exist any solution $\beta(s, v)$ of (21) such that $\alpha = \beta_v$.

Thus, for instance, since the function $\beta(s, v) = kv$ (k is a constant) is such that $\beta_v = k$, the subalgebra $\{V + S_k\}$ is conjugate to $\{V\}$. In this case, the associated invariant solution is found to be

$$W(s, v) = \tfrac{1}{3}K_0 s^3 + K_1 + kv, \tag{72}$$

which is simply a linear combination of solutions previously determined for subalgebras $\{V\}$ and $\{D\}$. In (t, x) space, we obtain the solution

$$\theta(t, x) = \frac{(x + k)^2}{2t} + \frac{t^3}{6K_0^2} + K_1, \tag{73}$$

which differs from the case found for V only by the addition of a constant to x.

For the general case, consider the subalgebra spanned by the generator

$$V + S_\alpha = \partial_v + \alpha(s, v)\partial_W. \tag{74}$$

A solution $W(s, v)$ invariant under this subalgebra must take the form

$$W(s, v) = \int \alpha(s, \xi)\,\mathrm{d}\xi + F(s), \tag{75}$$

and it is manifest that if such a solution exists, then $W_v = \alpha$, which implies that the generator (74) is conjugate to V.

A similar principle applies if we seek solutions invariant under a subalgebra of the form $D + aM + S_\alpha$, where $a \neq 0$ and α are solution of (21). The resulting solution will be fundamentally different from those previously found only if there does not exist a solution β of (21) such that

$$(a + 1)\beta - s\beta_s - v\beta_v = 0. \tag{76}$$

A solution $W(s, v)$ invariant under this subalgebra must take the form

$$W(s, v) = s^{a+1}\left(F\left(\frac{v}{s}\right) + \int \frac{\alpha(\sigma, v)}{\sigma^{a+2}}\,\mathrm{d}\sigma\right). \tag{77}$$

For the general case, it has not been determined which conditions apply on the function α in order to satisfy condition (76). In three specific instances, however, solutions have been obtained which are not linear combinations of those previously determined. Indeed, this allows us to establish a link with solutions already found for the subalgebras $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ [2].

(i) For $\beta = 1$, let us consider the subalgebra $\{D - M + c\varepsilon S_1\}$. The invariants are

$$\xi = \frac{v}{s}, \qquad W - (c\varepsilon)\ln(s), \tag{78}$$

so that the solution $W(s, v)$ of equation (21) is of the form

$$W(s, v) = F\left(\frac{v}{s}\right) + (c\varepsilon)\ln(s), \tag{79}$$

where F satisfies the following differential equation:

$$(1 - \xi^2)F_{\xi\xi} - 4\xi F_\xi + 3c\varepsilon = 0. \tag{80}$$

The solution is given by

$$W(s, v) = C_1\left(\frac{1}{4}\ln\left(\frac{v + s}{v - s}\right) - \frac{1}{2}\frac{vs}{v^2 - s^2}\right) + c\varepsilon\left(\frac{1}{2}\ln(v^2 - s^2) + \frac{s^2}{v^2 - s^2}\right) + C_2. \tag{81}$$

The change of variables (47) is difficult to invert in the general case, but in the specific case where $C_1 = 0$ we obtain

$$v(t,x) = -\frac{\varepsilon c x}{x^2 + 2\varepsilon c t}, \qquad s(t,x) = -\frac{c^2x^2 + 4\varepsilon c^3 t}{(x^2 + 2\varepsilon c t)^2}, \tag{82}$$

and the equivalent solution of the Chaplygin gas equation is thus

$$\theta(t,x) = \frac{1}{2}c\varepsilon \ln\left(\frac{2c^2}{x^2 + 2\varepsilon c t}\right) - \frac{3}{2}c\varepsilon + C_2. \tag{83}$$

This corresponds to a solution invariant under the subalgebra $\mathcal{K}_1$ of $\mathcal{G}$.

(ii) For $\beta = v$, let us take the subalgebra $\{D + c\varepsilon S_v\}$. The invariants are

$$\xi = \frac{v}{s}, \qquad \frac{W}{v} - (c\varepsilon)\ln(v), \tag{84}$$

so that the solution $W(s, v)$ of equation (21) is of the form

$$W(s,v) = vF\left(\frac{v}{s}\right) + (c\varepsilon) v \ln(v), \tag{85}$$

where F satisfies

$$(\xi^2 - \xi^4)F_{\xi\xi} + (2\xi - 4\xi^3)F_\xi + c\varepsilon = 0. \tag{86}$$

The solution is given by

$$W(s,v) = C_1\left(s + \frac{1}{2}v\ln\left(\frac{v-s}{v+s}\right)\right) + C_2 v + \frac{1}{2}c\varepsilon v \ln(v^2 - s^2). \tag{87}$$

In the specific case where $C_1 = 0$ we obtain

$$v(t,x) = \frac{\varepsilon t}{c}\,\mathbf{e}^{-\frac{2\varepsilon}{c}(x+C_2)}, \qquad s(t,x) = \sqrt{\frac{t^2}{c^2}\,\mathbf{e}^{-\frac{4\varepsilon}{c}(x+C_2)} - \mathbf{e}^{-\frac{2\varepsilon}{c}(x+C_2)}}, \tag{88}$$

and the equivalent solution of the Chaplygin gas equation is found to be

$$\theta(t,x) = -\tfrac{1}{2}t\,\mathbf{e}^{-\frac{2\varepsilon}{c}(x+C_2)}. \tag{89}$$

This corresponds to a solution invariant under the subalgebra $\mathcal{K}_2$ of $\mathcal{G}$.

(iii) For $\beta = s^2 - v^2$, let us take the subalgebra $\{D + M + c\varepsilon S_{(s^2-v^2)}\}$. The invariants are

$$\xi = \frac{v}{s}, \qquad \frac{W}{s^2 - v^2} - \frac{1}{2}(c\varepsilon)\ln(s^2 - v^2), \tag{90}$$

so that the solution $W(s, v)$ of equation (21) is of the form

$$W(s,v) = (s^2 - v^2)F\left(\frac{v}{s}\right) + \frac{1}{2}c\varepsilon(s^2 - v^2)\ln(s^2 - v^2), \tag{91}$$

where F satisfies

$$(\xi^4 - 2\xi^2 + 1)F_{\xi\xi} + (4\xi^3 - 4\xi)F_\xi - 2c\varepsilon = 0. \tag{92}$$

The solution is given by

$$\begin{aligned} W(s,v) = {} & C_1\left(\frac{1}{2}sv + \frac{1}{4}(s^2 - v^2)\ln\left(\frac{v+s}{v-s}\right)\right) + C_2(s^2 - v^2) \\ & + c\varepsilon\left(s^2 + \frac{1}{2}(s^2 - v^2)\ln(s^2 - v^2)\right). \end{aligned} \tag{93}$$

In the specific case where $C_1 = 0$ we obtain

$$v(t,x) = -\frac{\varepsilon x}{2c}, \qquad s(t,x) = \sqrt{\frac{x^2}{4c^2} + \mathbf{e}^{-\frac{\varepsilon}{c}(t+3c\varepsilon+2C_2)}}, \tag{94}$$

and the equivalent solution of the Chaplygin gas equation is therefore

Table 6. Fermionic potential $\psi(t,x)$ corresponding to the bosonic solutions.

Subalgebra	Solution $\psi(t,x)$
$\{V\}$	$\psi(t,x)=\phi\left(\frac{x}{t}+\frac{t}{K_0}\right)$
$\{D\}$	$\psi(t,x)=\phi\left(\frac{K_1}{2t}\left(1+\mathbf{e}^{-\frac{2}{K_1}(x+K_0)}\right)\right)$
$\{D-M\}$	$\psi(t,x)=\phi\left(\frac{4K_0x}{4K_0t+x^2}\right)$
$\{D+M\}$	$\psi(t,x)=\phi\left(\frac{2xK_1^{-1}}{\left(\mathbf{e}^{K_1^{-1}(t+2K_0)}-1\right)}\right)$
$\{D+2M\}$	$\psi(t,x)=\phi\left(\frac{t}{6K_0}+\frac{1}{6K_0}(K_1-2K_0)\frac{\sqrt{12K_0x-t^2}}{\sqrt{4K_0^2-K_1^2}}\right)$
$\left\{D+\frac{1}{2}M\right\}$	$\psi(t,x)=\phi\left(\frac{8x^2}{9K_0^2}+\frac{x}{t}-\frac{4x}{9K_0^2t}\sqrt{9K_0^2xt+4x^2t^2}\right)$
$\{D-M+c\varepsilon S_1\}$	$\psi(t,x)=\phi\left(\frac{4\varepsilon c^3t-2c^2xt+c^2x^2-\varepsilon cx^3}{(x^2+2\varepsilon ct)^2}\right)$
$\{D+c\varepsilon S_v\}$	$\psi(t,x)=\phi\left(\frac{\varepsilon t}{c}\mathbf{e}^{-\frac{2\varepsilon}{c}(x+C_2)}-\sqrt{\frac{t^2}{c^2}\mathbf{e}^{-\frac{4\varepsilon}{c}(x+C_2)}-\mathbf{e}^{-\frac{2\varepsilon}{c}(x+C_2)}}\right)$
$\{D+M+c\varepsilon S_{(s^2-v^2)}\}$	$\psi(t,x)=\phi\left(\frac{\varepsilon x}{2c}+\sqrt{\frac{x^2}{4c^2}+\mathbf{e}^{-\frac{\varepsilon}{c}(t+3c\varepsilon+2C_2)}}\right)$

$$\theta(t,x)=-\frac{\varepsilon}{4c}x^2-\frac{1}{2}c\varepsilon\,\mathbf{e}^{-\frac{\varepsilon}{c}(t+3c\varepsilon+2C_2)}. \tag{95}$$

This corresponds to a solution invariant under the subalgebra $\mathcal{K}_3$ of $\mathcal{G}$.

4.4. Solutions of the linear supersymmetric model

Solutions of the complete linear supersymmetric model can be determined from the solutions of the bosonic model described in the previous subsections. The fermionic scalar $\psi(t,x)$ is found through equation (27) by substituting the previously determined values of s and v as functions of t and x

$$\psi(t,x)=\phi(v-s), \tag{96}$$

where ϕ is an arbitrary function. The resulting functions for each of the cases which we were able to successfully invert in the subsections above are listed in table 6.

5. Extension to the (2 + 1)-dimensional case

5.1. Standard form of the planar model and invariant superalgebra

We now proceed to examine the supersymmetric planar model proposed by Jackiw and Polychronakos [10] and make a few observations concerning the possibility of extending our analysis to the (2 + 1)-dimensional case. Here, the velocity $\mathbf{v}$ of the fluid is supplemented by Grassmann (fermionic) variables ψ_a, $a=1,2$ that form a Majorana spinor ψ (real, two-component) [1]. The velocity is then

$$\mathbf{v}=\nabla\theta-\tfrac{1}{2}\psi^t\nabla\psi, \tag{97}$$

where θ is the bosonic scalar potential. The field equations which govern the motion of the supersymmetric fluid, and involve the potentials θ and ψ and the density of the fluid ρ, read as

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \theta_t + \mathbf{v} \cdot \nabla\theta = \frac{1}{2}v^2 + \frac{\lambda}{\rho^2} + \frac{\sqrt{2\lambda}}{2\rho}\psi^t\alpha \cdot \nabla\psi,$$
$$\psi_t + \mathbf{v} \cdot \nabla\psi = \frac{\sqrt{2\lambda}}{\rho}\alpha \cdot \nabla\psi, \tag{98}$$

where

$$\alpha = (\alpha_1, \alpha_2), \qquad \alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{99}$$

Combining the field equations (98) with (97), we obtain the following field equation for the velocity:

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla\left(\frac{\lambda}{\rho^2}\right) + \frac{\sqrt{2\lambda}}{\rho}(\nabla\psi)^t(\alpha \cdot \nabla\psi). \tag{100}$$

As in the one-dimensional case, a superalgebra of symmetries and supersymmetries of the field equations (98) has been given [1, 10]. The symmetry generators are associated with a nonrelativistic similitude algebra containing the six generators of the Galilean algebra in (2 + 1) dimensions; i.e. [7]

$$P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad P_2 = \partial_y,$$
$$B_1 = t\partial_x + x\partial_\theta, \qquad B_2 = t\partial_y + y\partial_\theta, \tag{101}$$
$$R = -y\partial_x + x\partial_y + \tfrac{1}{2}\psi_2\partial_{\psi_1} - \tfrac{1}{2}\psi_1\partial_{\psi_2},$$

along with the following dilations:

$$D_1 = 2t\partial_t + x\partial_x + y\partial_y + \rho\partial_\rho, \qquad D_2 = x\partial_x + y\partial_y + 2\theta\partial_\theta - \rho\partial_\rho + \psi_1\partial_{\psi_1} + \psi_2\partial_{\psi_2}. \tag{102}$$

In addition, we still have the potential shift symmetry $Z = \partial_\theta$. Moreover, the following supersymmetry transformations leave equations (98) invariant [1, 7]:

$$x' = x + \eta\alpha_1\psi, \qquad y' = y + \eta\alpha_2\psi, \qquad t' = t,$$
$$\rho' = \rho - (\eta\boldsymbol{\alpha} \cdot \nabla\psi)\rho \qquad \theta' = \theta + \frac{1}{2}(\eta\boldsymbol{\alpha}\psi) \cdot \mathbf{v} + \frac{\sqrt{2\lambda}}{2\rho}\eta\psi, \tag{103}$$
$$\psi' = \psi - (\mathbf{v} \cdot \boldsymbol{\alpha}\eta) - \frac{\sqrt{2\lambda}}{\rho}\eta,$$

and

$$x' = x, \qquad y' = y, \qquad t' = t, \qquad \rho' = \rho,$$
$$\theta' = \theta - \tfrac{1}{2}\tilde{\eta}\psi, \qquad \psi' = \psi - \tilde{\eta}, \tag{104}$$

where $\eta = (\eta^1, \eta^2)$ and $\tilde{\eta} = (\tilde{\eta}^1, \tilde{\eta}^2)$ correspond to fermionic constant parameters. The corresponding supersymmetric infinitesimal generators are identified as

$$\begin{aligned} Q_1 = {} & \psi_2\frac{\partial}{\partial x} + \psi_1\frac{\partial}{\partial y} - \rho(\psi_{2,x} + \psi_{1,y})\frac{\partial}{\partial\rho} \\ & + \left(\frac{1}{2}\theta_x\psi_2 - \frac{1}{4}\psi_1\psi_{1,x}\psi_2 + \frac{1}{2}\theta_y\psi_1 - \frac{1}{4}\psi_2\psi_{2,y}\psi_1 + \frac{\sqrt{2\lambda}}{2\rho}\psi_1\right)\frac{\partial}{\partial\theta} \\ & + \left(-\theta_y + \frac{1}{2}\psi\psi_y - \frac{\sqrt{2\lambda}}{\rho}\right)\frac{\partial}{\partial\psi_1} + \left(-\theta_x + \frac{1}{2}\psi\psi_x\right)\frac{\partial}{\partial\psi_2}, \end{aligned} \tag{105}$$

$$Q_2 = \psi_1 \frac{\partial}{\partial x} - \psi_2 \frac{\partial}{\partial y} - \rho(\psi_{1,x} - \psi_{2,y}) \frac{\partial}{\partial \rho} + \left(\frac{1}{2}\theta_x \psi_1 - \frac{1}{4}\psi_2 \psi_{2,x} \psi_1 - \frac{1}{2}\theta_y \psi_2 + \frac{1}{4}\psi_1 \psi_{1,y} \psi_2 + \frac{\sqrt{2\lambda}}{2\rho}\psi_2 \right) \frac{\partial}{\partial \theta} + \left(-\theta_x + \frac{1}{2}\psi \psi_x \right) \frac{\partial}{\partial \psi_1} + \left(\theta_y - \frac{1}{2}\psi \psi_y - \frac{\sqrt{2\lambda}}{\rho} \right) \frac{\partial}{\partial \psi_2}, \tag{106}$$

and

$$\tilde{Q}_1 = -\frac{1}{2}\psi_1 \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \psi_1}, \tag{107}$$

$$\tilde{Q}_2 = -\frac{1}{2}\psi_2 \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \psi_2}, \tag{108}$$

where Q_1 is associated with η^1, Q_2 with η^2, $\tilde{Q}_1$ with $\tilde{\eta}^1$ and $\tilde{Q}_2$ with $\tilde{\eta}^2$. The commutation relations between all the infinitesimal generators can be easily computed through the use of prolongation formulae (see chapters 2 and 5 of [8]).

5.2. *Transformed equation*

The desired result would be to find a system of transformed equations analogous to (21) and (26) in order to solve the field equations (98) through the use of a generalized Legendre transformation in two spatial dimensions. This has not yet been achieved for the general (2 + 1)-dimensional case, but a few elements of the theory can be discussed. In analogy with the transformation (14), we set

$$q = \theta_t - \tfrac{1}{2}\psi^t \psi_t, \qquad v_1 = \theta_x - \tfrac{1}{2}\psi^t \psi_x, \qquad v_2 = \theta_y - \tfrac{1}{2}\psi^t \psi_y, \tag{109}$$

and we postulate the existence of a function $\omega(q, v_1, v_2)$ such that

$$\theta(t, x, y) + \omega(q, v_1, v_2) = tq + xv_1 + yv_2. \tag{110}$$

As in the one-dimensional case, we once again introduce the sound speed $s = \sqrt{2\lambda}/\rho$. Making use of the field equations (98), we obtain the following relation between s, q, v_1 and v_2:

$$s^2 = 2q + v_1^2 + v_2^2. \tag{111}$$

Finally, if we consider ω as a function of s, v_1 and v_2 such that

$$\omega(q, v_1, v_2) = W(s, v_1, v_2), \tag{112}$$

the field equation for θ may be transformed to the following uniformly quadratic equation for W:

$$-3(W_s)^2 - 2sW_sW_{v_1v_1} - 2sW_sW_{v_2v_2} + 2sW_sW_{ss} + s^2\left(W_{v_1v_2}\right)^2 - s^2W_{v_1v_1}W_{v_2v_2} + s^2W_{v_1v_1}W_{ss} + s^2W_{v_2v_2}W_{ss} - s^2\left(W_{v_2s}\right)^2 - s^2\left(W_{v_1s}\right)^2 = 0. \tag{113}$$

The symmetry Lie algebra of this equation has not been fully determined, but the following generators constitute a partial list:

$$D = s\partial_s + v_1\partial_{v_1} + v_2\partial_{v_2} + W\partial_W, \qquad V_1 = \partial_{v_1}, \qquad V_2 = \partial_{v_2},$$
$$M = W\partial_W, \qquad R_v = -v_2\partial_{v_1} + v_1\partial_{v_2}. \tag{114}$$

The fermionic set of equations in (98) must also be written in the new variables (s, v_1, v_2). The explicit form is in fact very complicated and will not be relevant here. It is obtained by making the change of variables $(t, x, y) \leftrightarrow (s, v_1, v_2)$ in the set of equations:

$$\psi_{1,t} = -v_1\psi_{1,x} - (s - v_2)\psi_{1,y} + s\psi_{2,x}, \tag{115}$$

$$\psi_{2,t} = -v_1\psi_{2,x} - (s + v_2)\psi_{2,y} + s\psi_{1,x}. \tag{116}$$

It is important to mention here that in the new variables the transformed equations decouple and it is therefore reasonable to suppose that solutions of equation (113) could serve as the primary element in the construction of solutions of the planar supersymmetric model in the general case.

5.3. Invariant solutions of the supersymmetric planar model

The question arises as to whether it is possible to find solutions of the planar model which are invariant under the supersymmetric transformations themselves. Inspiring ourselves from the methods used in [11] for the super Korteweg–de Vries equation, we provide a partial answer to this question. Instead of using the supersymmetric generators (105)–(108) independently, it may be more to our advantage to consider a complex linear combination. For example, the combined generator $\tilde{Q} = \tilde{Q}_1 - \mathrm{i}\tilde{Q}_2$ has the property that $\tilde{Q}^2 = 0$, while the squares of the generators Q_1, Q_2, $\tilde{Q}_1$ and $\tilde{Q}_2$ are bosonic translation generators. A solution of the field equations (98) will be invariant under the $\tilde{Q}$ supersymmetry only if the fermionic fields ψ_1 and ψ_2 satisfy the condition $\psi_2 = -\mathrm{i}\psi_1$. Therefore, in order to impose $\tilde{Q}$ invariance, we are required to complexify the fermionic fields. Under these conditions, the bosonic fields θ and ρ will be preserved and we obtain the following simplifications:

$$\begin{aligned}\alpha\cdot\nabla\psi &= (\psi_{1,y} - \psi_{1,x})\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\\ &= -\mathrm{i}(\partial_x + \mathrm{i}\partial_y)\psi_1\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix} = -2\mathrm{i}(\partial_{\bar{z}}\psi_1)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix},\end{aligned} \tag{117}$$

using the change of variables $z = x + \mathrm{i}y$ (and $\bar{z} = x - \mathrm{i}y$). The field equations (98) now read as

$$\begin{aligned}&\rho_t + \nabla\cdot(\rho\nabla\theta) = 0,\\ &\theta_t + \frac{1}{2}(\nabla\theta)^2 = \frac{\lambda}{\rho^2} - \frac{2\sqrt{2\lambda}}{\rho}\mathrm{i}\psi_1(\partial_{\bar{z}}\psi_1),\\ &\psi_{1,t} + \nabla\theta\cdot\nabla\psi_1 = -\frac{2\sqrt{2\lambda}}{\rho}\mathrm{i}(\partial_{\bar{z}}\psi_1),\\ &\psi_{1,t} + \nabla\theta\cdot\nabla\psi_1 = +\frac{2\sqrt{2\lambda}}{\rho}\mathrm{i}(\partial_{\bar{z}}\psi_1).\end{aligned} \tag{118}$$

As a consequence of the last two equations, we get

$$\partial_{\bar{z}}\psi_1 = 0, \tag{119}$$

which implies that the fermionic potential is of the form $\psi_1 = \psi_1(z, t)$.

Therefore, given any solution $\theta(t, x, y)$, $\rho(t, x, y)$ of the bosonic planar Chaplygin gas model (where $\psi_1 = 0$), the first two equations of (118) will be satisfied identically. The last two equations of (118) imply that for any generalization of the bosonic solution to a solution of the supersymmetric planar model, the fermionic potential ψ_1 must satisfy the equation

$$\psi_{1,t} = -(\theta_x + \mathrm{i}\theta_y)\psi_{1,z}. \tag{120}$$

Finally, let us give some examples of solutions to the supersymmetric planar model constructed through the method described above. It is easy to show that the Legendre-transformed equation in two spatial equations, given in (113), admits in particular the symmetry subalgebra $\{V_1, V_2, R_v\}$ which consists of translation and rotation generators. Let us therefore consider solutions of equation (113) which are invariant under this subalgebra. The invariants are s and W, which implies a solution of the form $W = W(s)$. Equation (113) thus simplifies to the form

$$-3(W_s)^2 + 2sW_sW_{ss} = 0, \tag{121}$$

and leads to the invariant solution

$$W = \tfrac{2}{5}K_0 s^{5/2} + K_1, \tag{122}$$

which may be inverted through the two-dimensional Legendre transformation equations (110), (111) and (112) to give the bosonic solution

$$\theta(t, x, y) = \frac{x^2}{2t} + \frac{y^2}{2t} + \frac{1}{10K_0^4}t^5 + K_1, \qquad \rho(t, x, y) = \frac{\sqrt{2\lambda}K_0^2}{t^2}. \tag{123}$$

Solution (123) is invariant under the subalgebra $\{B_1, B_2, R\}$ in the original (t, x, y) space.

Postulating a solution of the form $\psi_2 = -\mathrm{i}\psi_1$, we apply equation (120) to the bosonic solution (123). The equation becomes

$$\psi_{1,t} = -\frac{z}{t}\psi_{1,z}, \tag{124}$$

which gives us the fermionic potentials

$$\psi_1(t, x, y) = \frac{C_0 t}{x + \mathrm{i}y}, \qquad \psi_2(t, x, y) = \frac{C_0 t}{\mathrm{i}x - y}. \tag{125}$$

Here, C_0 is a constant fermionic (odd-valued) Grassmann variable. Equations (123) and (125) together constitute a solution of the supersymmetric planar model.

Solutions can also be obtained using the field equations (98) directly. For instance, if we seek a solution which is invariant under the subsuperalgebra $\{D_1 - D_2, R, \tilde{Q}\}$, one possibility is

$$\theta(t, x, y) = -\frac{1}{2t}(x^2 + y^2), \qquad \rho(t, x, y) = \frac{\sqrt{\lambda}t}{\sqrt{x^2 + y^2}}. \tag{126}$$

Equation (120) then allows us to determine the fermionic functions

$$\psi_1(t, x, y) = C_1 t(x + \mathrm{i}y), \qquad \psi_2(t, x, y) = C_1 t(-\mathrm{i}x + y), \tag{127}$$

where C_1 is a fermionic constant.

Let us mention that the bosonic soliton solution

$$\theta(t, x, y) = \frac{\sinh^2(\alpha x + \beta y)}{2(\alpha^2 + \beta^2)t}, \qquad \rho(t, x, y) = \frac{\sqrt{2\lambda(\alpha^2 + \beta^2)t}}{\sinh^2(\alpha x + \beta y)}, \tag{128}$$

cannot be extended to a non-trivial supersymmetric solution since equation (120) is not compatible (i.e. x and y cannot be combined to obtain z).

6. Summary and concluding remarks

Through the use of a generalized Legendre transformation, a number of analytic solutions have been found for the supersymmetric Chaplygin gas in one spatial dimension. For this purpose, the subalgebra structure of the symmetry Lie algebra $\mathcal{L}$ was used to obtain group-invariant solutions of the transformed field equation (21). In many cases, these corresponded (via the Legendre transformation) to previously determined solutions of the standard field equations (2)–(4) which are invariant with respect to subalgebras of the Lie symmetry superalgebra $\mathcal{G}_s$, or generalizations of such solutions. In other cases, the solutions were completely new. Importantly, due to the very simple form of equation (26) for the field χ, the transformation from the (t, x) space to the (s, v) space allowed us to easily obtain the form of the fermionic potential ψ.

In addition, certain basic elements of a possible extension of our method to the supersymmetric planar model have been formulated. In particular, the (2 + 1)-dimensional analogue (113) of the transformed equation (21) has been determined, and some of its Lie point symmetries in (s, v_1, v_2) space have been identified. For the specific case where $\psi_2 = -\mathrm{i}\psi_1$, it has been shown how an invariant solution of (113) can be inverted through the Legendre transformation and then extended to a solution of the planar supersymmetric model. In the general case, it has not yet been determined how to incorporate the fermionic potentials ψ_1 and ψ_2 into the framework of a generalized Legendre transformation in (2 + 1) dimensions. In particular, the analogue of equation (26) for the planar case remains undiscovered.

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